5.3: Sum and Difference Identities

Sum and Difference Formulas:

$\sin(\alpha+\beta) =$ $\sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\sin(\alpha-\beta) =$ $\sin\alpha\cos\beta - \cos\alpha\sin\beta$	$\cos(\alpha+\beta) =$ $\cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\cos(\alpha-\beta) =$ $\cos\alpha\cos\beta + \sin\alpha\sin\beta$
$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$	

Remember that the sine sign STAYS, the cosine sign CHANGES, and the tangent does BOTH!

Example 1: Find the exact value of each expression.

a) $\cos(120° - 45°)$

$\cos 120 \cos 45 + \sin 120 \sin 45$

$\left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$-\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$

b) $\sin(60° - 45°)$

$\sin 60 \cos 45 - \cos 60 \sin 45$

$\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

c) $\sin\left(\frac{4\pi}{3} + \frac{\pi}{4}\right)$

$\sin\frac{4\pi}{3}\cos\frac{\pi}{4} + \cos\frac{4\pi}{3}\sin\frac{\pi}{4}$

$\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

d) $\sin(75°) = \sin(45° + 30°)$

$\sin 45 \cos 30 + \cos 45 \sin 30$

$\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

e) $\tan\left(\frac{5\pi}{3} - \frac{\pi}{4}\right)$

$\dfrac{\tan\frac{5\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{5\pi}{3}\tan\frac{\pi}{4}}$

$\dfrac{-\sqrt{3} - 1}{1 - \sqrt{3}} \cdot \dfrac{1+\sqrt{3}}{1+\sqrt{3}} = \dfrac{-4}{-2} - \dfrac{2\sqrt{3}}{-2} = +2 + \sqrt{3}$

f) $\tan\left(\frac{7\pi}{12}\right) = \tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \dfrac{\tan\frac{\pi}{4} + \tan\frac{\pi}{3}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{3}}$

$\dfrac{1+\sqrt{3}}{1-\sqrt{3}} \cdot \dfrac{1+\sqrt{3}}{1+\sqrt{3}} = \dfrac{1+2\sqrt{3}+3}{1-3} = \dfrac{4}{-2} + \dfrac{2\sqrt{3}}{-2}$

$= -2 - \sqrt{3}$

You Try:

a) $\sin(15^\circ)$ = SIN $(45-30)$

$\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

b) $\cos(165^\circ)$ COS $(120+45)$

$-\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$

c) $\sin\left(\frac{5\pi}{12}\right)$

SIN $\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$

$\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$

d) $\tan(-15^\circ)$

tan $(45-60)$

$-2+\sqrt{3}$

Working Backwards…

- **Write each expression as the sine, cosine, or tangent of an angle.**
- **Find the exact value of the expression**

Example 3: Write the expression as the sine, cosine, or tangent of a single angle. Then, evaluate if possible.

a) $\cos 50^\circ \cos 5^\circ + \sin 50^\circ \sin 5^\circ$

COS $(50^\circ - 5^\circ)$

COS (45°)

$\frac{\sqrt{2}}{2}$

b) $\sin 40^\circ \cos 20^\circ + \cos 40^\circ \sin 20^\circ$

SIN $(40^\circ + 20^\circ)$

SIN (60°)

$\frac{\sqrt{3}}{2}$

c) $\dfrac{\tan 50^\circ - \tan 20^\circ}{1+\tan 50^\circ \tan 20^\circ}$

tan$(50^\circ - 20^\circ)$

tan 30°

$\frac{\sqrt{3}}{3}$

d) $\sin\frac{7\pi}{12}\cos\frac{\pi}{12} - \cos\frac{7\pi}{12}\sin\frac{\pi}{12}$

SIN $\left(\frac{7\pi}{12} - \frac{\pi}{12}\right)$

SIN $\left(\frac{\pi}{2}\right)$

1

e) $\cos\frac{5\pi}{18}\cos\frac{\pi}{9}+\sin\frac{5\pi}{18}\sin\frac{\pi}{9}$

$\cos\left(\frac{5\pi}{18}-\frac{\pi}{9}\right)$

$\cos\left(\frac{\pi}{6}\right)$

$\frac{\sqrt{3}}{2}$

f) $\dfrac{\tan\frac{\pi}{5}+\tan\frac{4\pi}{5}}{1-\tan\frac{\pi}{5}\tan\frac{4\pi}{5}}$

$\tan\left(\frac{\pi}{5}+\frac{4\pi}{5}\right)$

$\tan(\pi)$

0

You Try

a) $\sin 27^\circ \cos 57^\circ - \sin 57^\circ \cos 27^\circ$

$= \sin(27^\circ - 57^\circ)$

$= \sin(-30^\circ)$ Q_4 ref. angle $= 30^\circ$

$= -\frac{1}{2}$

b) $\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{2}\right)-\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{2}\right)$

$\cos\left(\frac{\pi}{6}+\frac{\pi}{2}\right)$

$\cos\left(\frac{2\pi}{3}\right)$

$-\frac{1}{2}$

c) $\dfrac{\tan 20^\circ + \tan 34^\circ}{1-\tan 20^\circ \tan 34^\circ}$

$= \tan(20+34)$

$= \tan 54^\circ$

d) $\cos\frac{5\pi}{9}\cos\frac{\pi}{9}-\sin\frac{\pi}{9}\sin\frac{5\pi}{9}$

$= \cos\left(\frac{5\pi}{9}+\frac{\pi}{9}\right)$

$= \cos\frac{2\pi}{3}$ Q_2 ref. angle $= \frac{\pi}{3}$

$= -\frac{1}{2}$