Using Fundamental Identities

In Chapter 4, you studied the basic definitions, properties, graphs, and applications of the individual trig function.

Learning Targets
1. Evaluate trigonometric functions. 2. Simplify trigonometric functions. 3. Develop additional trig identities. 4. Solve trig equations.

- There are six main types of ***basic trigonometric identities.***
  - Reciprocal Identities (we know these already)
  - Quotient Identities (we have discussed these)
  - Pythagorean Identities
  - Co-function Identities
  - Negative Angle Identities
- The trigonometric identities express important fundamental relationships amongst the values of different trigonometric function.
- These identities are useful whenever expressions involving trigonometric functions need to be simplified. This skill is of supreme importance in the study of Calculus.

I. Some trigonometric identities follow directly from the definitions of the six basic trig functions. These *basic identities* consist of the *reciprocal identities* and the *quotient identities.*

RECIPROCAL IDENTITIES		
$\sin\theta = \dfrac{1}{\csc\theta}$	$\cos\theta = \dfrac{1}{\sec\theta}$	$\tan\theta = \dfrac{1}{\cot\theta}$
$\csc\theta = \dfrac{1}{\sin\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cot\theta = \dfrac{1}{\tan\theta}$

QUOTIENT IDENTITIES	
$\dfrac{\sin\theta}{\cos\theta} = \tan\theta$	$\dfrac{\cos\theta}{\sin\theta} = \cot\theta$

PYTHAGOREAN IDENTITIES		
$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$1 + \cot^2\theta = \csc^2\theta$

Example 1: Using Identities

a. Find $\sin\theta$ and $\cos\theta$ if $\tan\theta = 5$ and $\cos\theta > 0$

SOLUTION We could solve this problem by the reference triangle techniques of Section 4.3 (see Example 7 in that section), but we will show an alternate solution here using only identities.

First, we note that $\sec^2\theta = 1 + \tan^2\theta = 1 + 5^2 = 26$, so $\sec\theta = \pm\sqrt{26}$. Since $\sec\theta = \pm\sqrt{26}$, we have $\cos\theta = 1/\sec\theta = 1/\pm\sqrt{26}$. But $\cos\theta > 0$, so $\cos\theta = 1/\sqrt{26}$.

Finally,

$$\tan\theta = 5$$

$$\frac{\sin\theta}{\cos\theta} = 5$$

$$\sin\theta = 5\cos\theta = 5\left(\frac{1}{\sqrt{26}}\right) = \frac{5}{\sqrt{26}}$$

Therefore, $\sin\theta = \dfrac{5}{\sqrt{26}}$ and $\cos\theta = \dfrac{1}{\sqrt{26}}$.

Now try Exercise 1.

b. If $\cos\theta = -\dfrac{3}{10}$, use trig identities to find the other five trig function values of θ. Assume $\dfrac{\pi}{2} < \theta < \pi$.

$$\sin^2\theta + \cos\theta = 1$$

$$\sin^2\theta + \frac{9}{100} = 1$$

$$\sin^2\theta = \frac{91}{100}$$

$$\sin\theta = \frac{\sqrt{91}}{10}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = \frac{\frac{\sqrt{91}}{100}}{\frac{-3}{10}}$$

$$\tan\theta = \left(\frac{\sqrt{91}}{100}\right)\left(\frac{10}{-3}\right)$$

$$\tan\theta = \frac{-\sqrt{91}}{3}$$

$$\csc\theta = \frac{10}{\sqrt{91}}$$

$$\sec\theta = -\frac{10}{3}$$

$$\cot\theta = -\frac{3}{\sqrt{91}}$$

c. Find $\sec\theta$ and $\csc\theta$ if $\tan\theta = 3$ and $\cos\theta > 0$.

$sec^2\theta = 1 + tan^2\theta$ $sec^2\theta = 1 + 3^2$ $sec^2\theta = 10$ $sec\theta = \sqrt{10}$	Using a trigonometric identity property, plugging in the given values, and solving.
$\dfrac{sec\theta}{csc\theta} = tan\theta$ $csc\theta = \dfrac{\sqrt{10}}{3}$	Using a trigonometric equivalent for tan and plugging in already known values to solve for sin.

d. Use the values $\sec u = -\dfrac{3}{2}$ and $\tan\theta > 0$ to find the values of all six trig functions

Use the values $\sec u = -\frac{3}{2}$ and $\tan u > 0$ to find the values of all six trigonometric functions.

Solution

Using a reciprocal identity, you have

$$\cos u = \frac{1}{\sec u} = \frac{1}{-3/2} = -\frac{2}{3}.$$

Using a Pythagorean identity, you have

$$\begin{aligned}\sin^2 u &= 1 - \cos^2 u && \text{Pythagorean identity}\\ &= 1 - \left(-\frac{2}{3}\right)^2 && \text{Substitute } -\tfrac{2}{3} \text{ for } \cos u.\\ &= 1 - \frac{4}{9} = \frac{5}{9}. && \text{Simplify.}\end{aligned}$$

Because $\sec u < 0$ and $\tan u > 0$, it follows that u lies in Quadrant III. Moreover, because $\sin u$ is negative when u is in Quadrant III, you can choose the negative root and obtain $\sin u = -\sqrt{5}/3$. Now, knowing the values of the sine and cosine, you can find the values of all six trigonometric functions.

$$\sin u = -\frac{\sqrt{5}}{3} \qquad \csc u = \frac{1}{\sin u} = -\frac{3}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

$$\cos u = -\frac{2}{3} \qquad \sec u = \frac{1}{\cos u} = -\frac{3}{2}$$

EVEN/ODD IDENTITIES

Recall that a function f is ***even*** if, for every x in the domain of f, $f(-x) = f(x)$ and ***odd*** if $f(-x) = -f(x)$. Each of the six basic trigonometric functions is either odd or even.

NEGATIVE ANGLE IDENTITIES (EVEN/ODD IDENTITIES)		
$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\csc(-\theta) = -\csc\theta$	$\sec(-\theta) = \sec\theta$	$\cot(-\theta) = -\cot\theta$

COFUNCTION IDENTITIES

A trigonometric function f is a co-function of another trigonometric function g if $f(\alpha) = g(\beta)$ where α and β are complementary angles. Thus the "c-o" in co-function is short for complementary.

CO-FUNCTION IDENTITIES		
$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$	$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$
$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$
Note: to rewrite the identities in degree measure replace $\frac{\pi}{2}$ with 90°.		

Example 2: Using More Identities

a. $\cos\left(\frac{\pi}{2}-\theta\right)=\frac{3}{5}, \quad \cos\theta=\frac{4}{5}$

$*\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$ therefore $\sin\theta=\frac{3}{5}$ and $\csc\theta=\frac{5}{3}$

$\cos\theta=\frac{4}{5}$ so $\sec\theta=\frac{5}{4}$

$\tan\theta=\frac{\sin\theta}{\cos\theta} = \frac{\frac{3}{5}}{\frac{4}{5}} = \left(\frac{3}{5}\right)\left(\frac{5}{4}\right) = \frac{3}{4}$ and $\cot\theta=\frac{4}{3}$

b. If $\cos\theta=0.34$, find $\sin\left(\theta-\frac{\pi}{2}\right)$

SOLUTION This problem can best be solved using identities.

$$\sin\left(\theta-\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}-\theta\right) \quad \text{Sine is odd.}$$
$$= -\cos\theta \quad \text{Cofunction identity}$$
$$= -0.34$$

Now try Exercise 7.

II. Simplifying Trig Expressions.

In calculus it is often necessary to deal with expressions that involve trig function. Some of those expressions start out looking fairly complicated, but it is often possible to use identities along with algebraic techniques to simplify the expressions before dealing with them.

Example 3: Factoring a Trig Expression

a. $\sec^2\theta-1$

Difference of Sqs

$(\sec\theta-1)(\sec\theta+1)$

b. $4\tan^2 x+\tan x-3$

FACTOR BY GROUPING

$4\tan^2 x+4\tan x-3\tan x-3$

$4\tan x(\tan x+1)-3(\tan x+1)$

$(4\tan x-3)(\tan x+1)$

c. $\csc^2\theta - \cot\theta - 3$

FACTOR BY GROUPING

$4\tan^2 x + 4\tan x - 3\tan x - 3$

$4\tan x(\tan x + 1) - 3(\tan x + 1)$

$(4\tan x - 3)(\tan x + 1)$

d. $\dfrac{\cos^2 x - 4}{\cos x - 2}$

$\dfrac{(\cos x - 2)(\cos x + 2)}{\cos x - 2}$ ← DOS

$\cos x + 2$ Divide

Example 4: Simplify to a Single Trig Function

a. Simplify the expression $\sin^3 x + \sin x \cos^2 x$

SOLUTION

Solve Algebraically

$$\begin{aligned} \sin^3 x + \sin x \cos^2 x &= \sin x\,(\sin^2 x + \cos^2 x) \\ &= \sin x\,(1) && \text{Pythagorean identity} \\ &= \sin x \end{aligned}$$

b. Simplify the expression $\sin x \cos^2 x - \sin x$

Solution

First factor out a common monomial factor and then use a fundamental identity.

$$\begin{aligned} \sin x \cos^2 x - \sin x &= \sin x(\cos^2 x - 1) && \text{Factor out common monomial factor.} \\ &= -\sin x(1 - \cos^2 x) && \text{Factor out } -1. \\ &= -\sin x(\sin^2 x) && \text{Pythagorean identity} \\ &= -\sin^3 x && \text{Multiply.} \end{aligned}$$

c. Simplify $\sin t + \cot t \cos t$

Solution

Begin by rewriting cot t in terms of sine and cosine.

$$\sin t + \cot t \cos t = \sin t + \left(\frac{\cos t}{\sin t}\right)\cos t \qquad \text{Quotient identity}$$

$$= \frac{\sin^2 t + \cos^2 t}{\sin t} \qquad \text{Add fractions.}$$

$$= \frac{1}{\sin t} \qquad \text{Pythagorean identity}$$

$$= \csc t \qquad \text{Reciprocal identity}$$

d. Simplify $\dfrac{(\sec x + 1)(\sec x - 1)}{\sin^2 x}$

$$\frac{(\sec x + 1)(\sec x - 1)}{\sin^2 x} = \frac{\sec^2 x - 1}{\sin^2 x} \qquad (a + b)(a - b) = a^2 - b^2$$

$$= \frac{\tan^2 x}{\sin^2 x} \qquad \text{Pythagorean identity}$$

$$= \frac{\sin^2 x}{\cos^2 x} \cdot \frac{1}{\sin^2 x} \qquad \tan x = \frac{\sin x}{\cos x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

e. Simplify: $\dfrac{\cos x}{1-\sin x}-\dfrac{\sin x}{\cos x}$

SOLUTION

$$\frac{\cos x}{1-\sin x}-\frac{\sin x}{\cos x}$$

$$=\frac{\cos x}{1-\sin x}\cdot\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}\cdot\frac{1-\sin x}{1-\sin x}$$ Rewrite using common denominator.

$$=\frac{(\cos x)(\cos x)-(\sin x)(1-\sin x)}{(1-\sin x)(\cos x)}$$

$$=\frac{\cos^2 x-\sin x+\sin^2 x}{(1-\sin x)(\cos x)}$$

$$=\frac{1-\sin x}{(1-\sin x)(\cos x)}$$ Pythagorean identity

$$=\frac{1}{\cos x}$$

$$=\sec x$$

f. Simplify: $\dfrac{\sin x}{1+\cos x}+\dfrac{\cos x}{\sin x}$

Solution

$$\frac{\sin\theta}{1+\cos\theta}+\frac{\cos\theta}{\sin\theta}=\frac{(\sin\theta)(\sin\theta)+(\cos\theta)(1+\cos\theta)}{(1+\cos\theta)(\sin\theta)}$$

$$=\frac{\sin^2\theta+\cos^2\theta+\cos\theta}{(1+\cos\theta)(\sin\theta)}$$ Multiply.

$$=\frac{\cancel{1+\cos\theta}}{\cancel{(1+\cos\theta)}(\sin\theta)}$$ Pythagorean identity: $\sin^2\theta+\cos^2\theta=1$

$$=\frac{1}{\sin\theta}$$ Divide out common factor.

$$=\csc\theta$$ Reciprocal identity