EXTREME VALUES OF FUNCTIONS and APPLICATIONS OF THE FIRST AND SECOND DERIVATIVE

I. Prerequisite Skills

Example: Find the x- and y-intercepts of the function.

a. $f(x) = 8x^2 - 10x + 3$

b. $f(x) = x^4 - 5x^2 - 36$

II. First Derivative Test for Critical Points

Now that we know how to find some derivatives using the power and chain rule, we can talk about what information the first derivative of the function gives us about the function.

Definition: Let f be defined at c. If $f'(c) = 0$ or f' is undefined at c, then c is a **critical point** of f.

Example 1: Find the critical points of the following functions.

a. $f(x) = 3x^3 - 9x + 5$

b. $f(x) = 2x^3 - 15x^2 + 24x + 7$

c. $f(x) = 3x^3 - 18x^2 - 4$

Once we know a function's critical points, we can test them to determine whether it is a max or min (or neither).

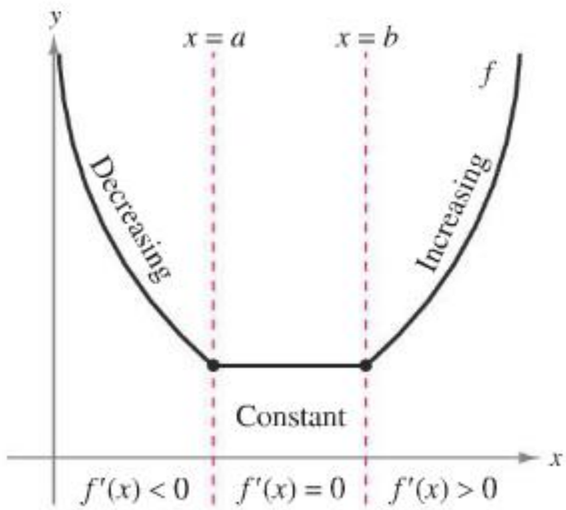
Because the first derivative represents the slope of a function, it can tell us information about when a function is increasing and decreasing. The following theorem summarizes what is meant by that statement.

THEOREM 3.5 **Test for Increasing and Decreasing Functions**

Let f be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b)

1. If $f'(x)>0$ for all x in (a,b), then f is increasing on $[a,b]$.
2. If $f'(x)<0$ for all x in (a,b), then f is decreasing on $[a,b]$.

The following picture illustrates this idea.



The derivative is related to the slope of a function.

This leads to another beautiful idea. It is called the First Derivative Test.

THEOREM 3.6 **The First Derivative Test**

Let c be a critical point of a function f that is continuous on an open interval containing c.

1. If $f'(x)$ changes from negative to positive at c, then f has a *relative minimum* at $x = c$.
2. If $f'(x)$ changes from positive to negative at c, then f has a *relative maximum* at $x = c$.
3. If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative maximum nor a relative minimum.

Steps to Finding Intervals of Increase and Decrease:

1. Find the critical points of f, and use these numbers to create test intervals (draw number line).

2. Determine the sign of $f'(x)$ at one test point in each of the intervals.

3. Use **Theorem 3.5** to determine whether f is increasing or decreasing on each interval and **Theorem 3.6** to determine if there is a max/min/neither

Example 2. For each of the following functions:

(a) Find the critical points.

(b) Determine the intervals on which the function is increasing/decreasing.

(c) Classify each critical point as a local max, local min, or neither.

a. $f(x) = x^3 - 27x - 20$

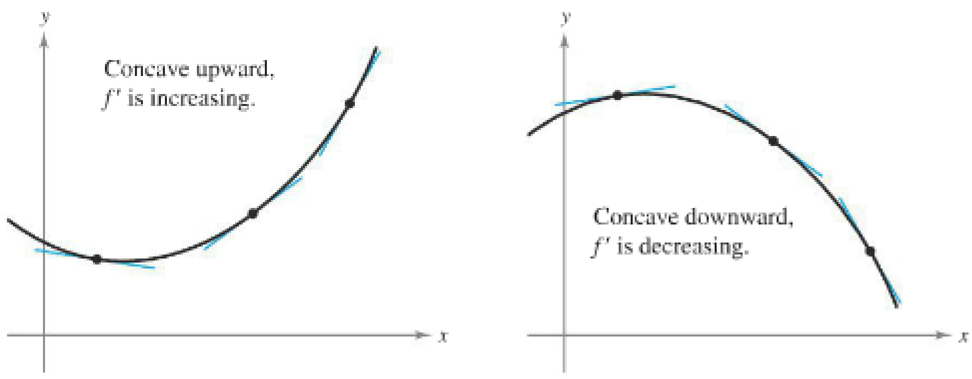
b. $f(x) = x^3 - \frac{3}{2}x^2$

c. $g(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 70x + 5$

d. $f(x) = \frac{1}{3}x^3 - x^2 + x$

III. Concavity and Points of Inflection

The second derivative of a function (which is the derivative of the derivative) tells us about the concavity of a function. If the graph of f is concave up, then the tangent lines are all below the function. If the graph of f is concave down, then the tangent lines are above the function.



(a) The graph of f lies above its tangent lines. **(b)** The graph of f lies below its tangent lines.

THEOREM 3.7 **Test for Concavity**

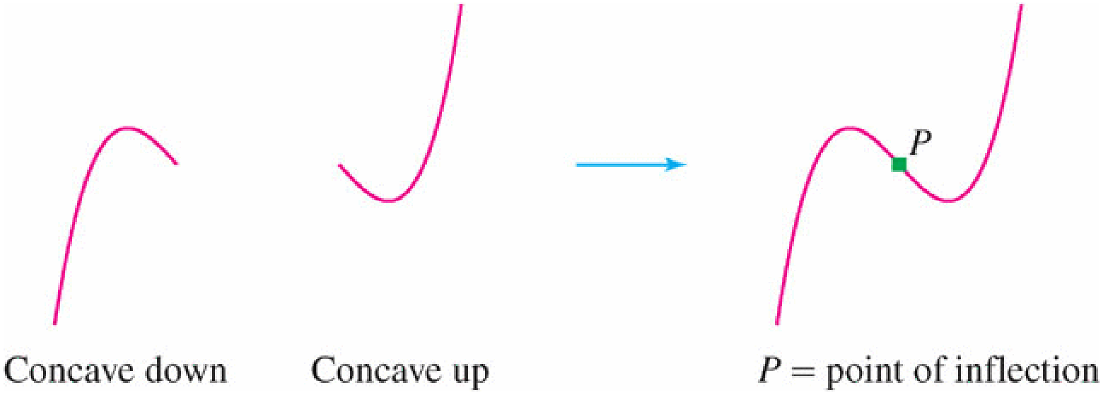
Let f be a function whose second derivative exists on an open interval.

1. If $f''(x) > 0$ for all x in the open interval, then the graph of f is concave up.

2. If $f''(x) < 0$ for all x in the open interval, then the graph of f is concave down.

We classify points at which a function changes concavity as **points of inflection**.

Definition: We say that $(c, f(c))$ is a **point of inflection** if $f''(c)=0$ and $f''(x)$ changes signs at $x=c$. This means that the function $f(x)$ goes from concave up to concave down or vice versa at $x=c$.



Example 3. Find the points of inflection and intervals of concavity of the following functions.

a. $f(x)=3x^5-5x^4+1$

b. $f(x)=x^4-4x^3$